were obtained from Fig. 1.4, Ref. 4, while those for the ellipses were obtained by Wiland[6] at $Re_\infty = 6.8 \times 10^4$. Wiland's results, after being corrected as indicated on page 60 of his report are: $(t/c) = 0.8$, $c_d = 0.67$; $(t/c) = 0.6$, $c_d = 0.40$. These test results are compared to the theoretical values: $(t/c) = 0.8$, $c_d = 0.71$; $(t/c) = 0.6$, $c_d = 0.34$. Thus the percentage deviation of the theoretical value from the experimental value is as follows. For $(t/c) = 0.8$, percentage deviation $= 6\%$; for $(t/c) = 0.6$, percentage deviation $= 15\%$. Details of the analysis, the computer program, and additional results are contained in Ref. 7.

In conclusion, the method presented here for the laminar separated flow over nonlifting ellipses appears to be in reasonable agreement with the available experimental data especially for the drag coefficient.

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Galerkin Finite Element Solution for the Stability of Cantilever Columns Subjected to Tangential Loads

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Introduction

STABILITY of nonconservative systems has been well discussed in the works of Bolotin[1] and Leipholz.[2] Recently, the finite element method has been successfully applied to the stability analysis of nonconservative systems. Barsoum[3] developed a finite element formulation based on the variational approach, whereas Kikuchi[4] presented a finite element formulation based on Mikhlin's[5] work on the Galerkin method. It is interesting to note that both the above formulations[3,4] gave identical element matrices for the stability analysis of cantilever column with a concentrated tangential load at the free end. In this Note, a different finite element formulation based on the weighted residual method using the Galerkin criterion is presented. Three problems concerning the stability of uniform cantilever column subjected to tangential loads are presented. The results indicate that the convergence of the present finite element method is excellent and even with a two-element idealization, very accurate results are obtained.

Received September 5, 1974. The authors take this opportunity to thank C. L. Amba-Rao for his constant encouragement throughout the course of this work.

Index category: Structural Stability Analysis.

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Finite Element Formulation

The differential equation governing the lateral motion of a column with tangential load, in the nondimensional form, is given by ($x = 0$ is fixed end and $x = 1$ is the free end).

$$W'''' + \lambda f(x) W'' - \Omega W = 0 \tag{1}$$

where $'$ denotes differentiation with respect to x. Here, three cases are considered:

Case 1: Subtangential concentrated load P at the free end for which $\lambda = PL^2/EI$ and $f(x) = 1$, where L is the length of the column, E is the Young's modulus and I is the moment of inertia.

Case 2: A uniformly distributed tangential load of intensity q_o per unit length, for which $\lambda = q_o L^3/EI$ and $f(x) = (1-x)$.

Case 3: A triangularly varying tangential load q of intensity $q = q_o(1-x)$, for which

$$\lambda = q_o L^4/EI \qquad \text{and} \qquad f(x) = \tfrac{1}{2}(1-x)^2$$

Ω is the frequency parameter defined as $\Omega = m\omega^2 L^4/EI$, where m is the mass of the column per unit length and ω is the circular frequency.

The boundary conditions are:

$$\text{1) at} \quad x = 0 \quad W = 0 \quad \text{and} \quad W' = 0 \tag{2}$$

$$\text{2) at} \quad x = 1 \quad W'' = 0 \quad \text{and} \quad W''' - \lambda(\gamma - 1)W' = 0 \quad \text{(for case 1)} \tag{3}$$

(Here $\gamma = 0$ represents Euler column and $\gamma = 1$ represents Beck's[6] column) $W''' = 0$ (for cases 2 and 3).

In the present finite element formulation, the domain of the column is subdivided into a number of elements. A seventh degree polynomial in x is assumed over each element as

$$W_e = [1 \; x \; x^2 \; x^3 \; x^4 \; x^5 \; x^6 \; x^7]\{\alpha\}_e \tag{4}$$

where

$$\{\alpha\}_e{}^T = [\alpha_1 \; \alpha_2 \; \alpha_3 \; \alpha_4 \; \alpha_5 \; \alpha_6 \; \alpha_7 \; \alpha_8] \tag{5}$$

Equation (4) is concisely written as

$$W_e = [F]\{\alpha\}_e \tag{6}$$

The eight undetermined coefficients α_1–α_8 are determined by the nodal parameters W, W', W'', W''' at each end of the element as

$$\{\delta\}_e = [T]\{\alpha\}_e \tag{7}$$

where

$$\{\delta\}_e{}^T = [W_1 \; W_1' \; W_1'' \; W_1''' \; W_2 \; W_2' \; W_2'' \; W_2'''] \tag{8}$$

From Eqs. (6) and (7), we obtain

$$W_e = [\phi]\{\delta\}_e \tag{9}$$

where

$$[\phi] = [F][T]^{-1} \tag{10}$$

Substituting Eq. (9) in Eq. (1) we get the residual R_e for the element as

$$R_e = [\phi'''']\{\delta\}_e + \lambda f(x)[\phi'']\{\delta\}_e - \Omega[\phi]\{\delta\}_e \tag{11}$$

In the Galerkin finite element method, the weighted residual is minimized as

$$\frac{\partial}{\partial\{\delta\}_e} \int W_e \cdot R_e \, dx = 0 \tag{12}$$

The process is repeated for all the elements and after the usual assembly procedure, one gets the matrix equation as

$$[K]\{\delta\} + \lambda[K_G]\{\delta\} - \Omega[M]\{\delta\} = 0 \tag{13}$$

where $\{\delta\}$ is the vector of nodal displacements of the structure.

The critical load λ_{cr} is obtained from Eq. (13) following the well known dynamic criterion where the two lowest frequencies just degenerate. It is to be noted here that the boundary conditions represented in Eqs. (2) and (3) can be easily satisfied for cases (2) and (3) in the present formulation. For case (1), a congruent transformation has to be effected so that $[W''' - \lambda(\gamma - 1)W']$ will be one of the nodal parameters and then can be easily satisfied in the usual manner.

Numerical Examples and Discussion

Using the present Galerkin finite element method, the critical loads for three cases of stability of uniform cantilever column subjected to tangential loads, namely: 1) a subtangential concentrated load P at the free end; 2) a uniformly distributed tangential load of intensity q_o per unit length; and 3) a triangularly varying distributed tangential load of intensity q, where $q = q_o(1-x)$ are obtained.

Table 1 gives the critical load parameter λ_{cr} for case (1), for various values of γ. For the sake of comparison, the results of Ref. 4 and Beck's[6] values for $\gamma = 1$ are included in this Table. The results indicate that the present results are very accurate even with a two element idealization of the column.

Table 1 Stability parameter λ_{cr} of a uniform cantilever column subjected to sub-tangential concentrated load at the free end

γ	Present Work 1 element[a]	Present Work 2 elements[b]	Ref. 4 5 elements[c]	Beck's[6] value
0.7	16.79019	16.78750	16.80	...
0.8	17.59646	17.58923	17.60	...
0.9	18.68248	18.66848	18.68	...
1.0	20.07538	20.05102	20.07	20.05

[a] Order of the dynamical matrix 4. [b] Order of the dynamical matrix 8.
[c] Order of the dynamical matrix 10.

Table 2 gives the critical load parameter λ_{cr} for cases (2) and (3) for one and two element idealizations. The results of Ref. 2 are also included for comparison. It can be seen from Table 2 that there is a slight discrepancy between the present results and those of Ref. 2. Because of the high accuracy and rapid convergence of the present finite element, we believe that the results obtained by the present element are more accurate.

Table 2 Stability parameter λ_{cr} of a uniform cantilever column with distributed tangential load

No. of elements	Case 1 Present work	Case 1 Ref. 2	Case 2 Present work	Case 2 Ref. 2
1[a]	40.05385		150.90001	
2[b]	40.05376		150.64225	
		40.7		158.2

[a] Order of the dynamical matrix 4. [b] Order of the dynamical matrix 8.

Conclusions

A Galerkin finite element is developed and successfully applied for the stability analysis of uniform cantilever column subjected to three types of tangential loading conditions. The results indicate that the convergence is very rapid and accurate results can be obtained even with a two element idealization. It is worthwhile to apply the method to other problems of stability of nonconservative systems and test the reliability of the present Galerkin finite element method.

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Radiation from an Array of Longitudinal Fins of Triangular Profile

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Nomenclature

A_P = profile area
A_R = reference area, $\sigma\varepsilon r_i^3 T_b^3/\kappa$
κ = thermal conductivity
L = fin width
N = number of node points
Q = heat transfer per unit length of fin array
Q_1 = heat transfer per unit length of fin array from Ref. 3
r = radial distance
r_i = tube radius
r_o = radial distance to the fin tip
t = local fin thickness
t_i = thickness of fin at the root
t_o = thickness of fin at the tip
T = temperature
T_b = base temperature
ε = emissivity
σ = Stefan-Boltzmann constant

Dimensionless Parameters

N_c = $2\sigma\varepsilon T_b^3 r_i^2/\kappa t_i$ (the conduction number)
N_f = the number of fins
N_L = L/r_i
N_P = t_o/t_i
Q^* = $Q/(2\pi r_i \sigma T_b^4)$

Introduction

HEAT rejection in space depends solely on thermal radiation. This has resulted in widespread interest in radiation from various kinds of fins and fin arrays. One problem of particular interest is the augmentation of heat transfer from the outside surface of a circular tube by the addition of fins.

The circular fin array has been analyzed by Sparrow et al.[1] for black fins of rectangular profile and by Schnurr and Cothran[2] for gray circular fins of triangular profile. The straight fin and tube array has been investigated by Karlekar and Chao.[3] Their analysis included fin to fin interactions but fin to base interactions were neglected. Their results should, therefore, be applied only to cases where the tube radius is much smaller than the fin width.

The purpose of the work reported here is to analyze the straight fin and tube array (Fig. 1) including the effects of both fin to fin and fin to base interactions and to present results which may be used to optimize the design with respect to weight.

Received October 4, 1974. The author gratefully acknowledges the support of the NASA-Marshall Space Flight Center in the early stages of this work and particularly the aid and encouragement of C. A. Cothran.
Index category: Radiation and Radiative Heat Transfer.
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